are

$$N_{xx} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{n\pi}{b}\right)^2\left[B_{yx}^*\left(\frac{m\pi}{a}\right)^4 + (B_{xx}^* + B_{yy}^* - 2B_{ss}^*)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + B_{xy}^*\left(\frac{n\pi}{b}\right)^4\right]\Phi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$N_{yy} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m\pi}{a}\right)^2\left[B_{yx}^*\left(\frac{m\pi}{a}\right)^4 + (B_{xx}^* + B_{yy}^* - 2B_{ss}^*)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + B_{xy}^*\left(\frac{n\pi}{b}\right)^4\right]\Phi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$N_{xy} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\pi}{a}\frac{n\pi}{b}\left[B_{yx}^*\left(\frac{m\pi}{a}\right)^4 + (B_{xx}^* + B_{yy}^* - 2B_{ss}^*)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + B_{xy}^*\left(\frac{n\pi}{b}\right)^4\right]\Phi_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$\begin{aligned} M_{xx} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Bigg\{&(B_{yx}^{*2} + A_{yy}^*D_{xx}^*)\left(\frac{m\pi}{a}\right)^6 + [B_{yx}^*(B_{yy}^* + 2B_{xx}^* - 2B_{ss}^*) + D_{xx}^*(2A_{xy}^* + A_{ss}^*) + D_{xy}^*A_{yy}^*]\left(\frac{m\pi}{a}\right)^4\left(\frac{n\pi}{b}\right)^2 + \\ &[B_{xx}^*(B_{xx}^* + B_{yy}^* - 2B_{ss}^*) + B_{yx}^*B_{xy}^* + A_{xx}^*D_{xx}^* + D_{xy}^*(2A_{xy}^* + A_{ss}^*)]\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^4 + \\ &(B_{xx}^*B_{xy}^* + A_{xx}^*D_{xy}^*)\left(\frac{n\pi}{b}\right)^6\Bigg\}\Phi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \end{aligned}$$

$$\begin{aligned} M_{yy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Bigg\{&(B_{yy}^*B_{yx}^* + A_{yy}^*D_{xy}^*)\left(\frac{m\pi}{a}\right)^6 + [B_{xy}^*B_{yx}^* + B_{yy}^*(B_{xx}^* + B_{yy}^* - 2B_{ss}^*) + D_{xy}^*(2A_{xy}^* + A_{ss}^*) + A_{yy}^*D_{yy}^*]\left(\frac{m\pi}{a}\right)^4\left(\frac{n\pi}{b}\right)^2 + \\ &[B_{xy}^*(B_{xx}^* + 2B_{yy}^* - 2B_{ss}^*) + A_{xx}^*D_{xy}^* + D_{yy}^*(2A_{xy}^* + A_{ss}^*)]\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^4 + \\ &(B_{xy}^{*2} + A_{xx}^*D_{yy}^*)\left(\frac{n\pi}{b}\right)^6\Bigg\}\Phi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \end{aligned} \tag{15}$$

$$\begin{aligned} M_{xy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\pi}{a}\frac{n\pi}{b}\Bigg\{&(B_{ss}^*B_{yx}^* - 2A_{yy}^*D_{ss}^*)\left(\frac{m\pi}{a}\right)^4 + [B_{ss}^*(B_{xx}^* + B_{yy}^* - 2B_{ss}^*) - 2D_{ss}^*(2A_{xy}^* + A_{ss}^*)]\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \\ &(B_{ss}^*B_{xy}^* - 2A_{xx}^*D_{xx}^*)\left(\frac{n\pi}{b}\right)^4\Bigg\}\Phi_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \end{aligned}$$

For a uniformly loaded plate ($p = p_0 =$ constant) the Fourier coefficients for the load are

$$\begin{aligned} p_{mn} &= 16p_o/(\pi^2 mn) \quad \text{for } m, n = \text{odd} \\ p_{mn} &= 0 \quad \text{for all other } m \text{ and } n \end{aligned} \tag{16}$$

and the deflection is given by

$$w = \frac{16p_o b^4}{\pi^6(D_{xx}^*D_{yy}^*)^{1/2}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\{w_{mn}\}}{[mn\{\Phi_{mn}\}]}\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{17}$$

where

$$\begin{aligned} \{w_{mn}\} &= \alpha^{-1}(m/c)^4 + \gamma n^2(m/c)^2 + \alpha n^4 \\ \{\Phi_{mn}\} &= (m/c)^8(\beta/\alpha + \omega^2) + n^2(m/c)^6(\delta/\alpha + \gamma\beta + 2\chi\omega) + n^4(m/c)^4(\alpha^{-1}\beta^{-1} + \alpha\beta + \gamma\delta + 2\omega\psi + \chi^2) + \\ &\quad n^6(m/c)^2(\gamma/\beta + \alpha\delta + 2\psi\chi) + n^8(\alpha/\beta + \psi^2) \\ c &= a/b \end{aligned}$$

Solutions for other loadings can be obtained by expanding the loading in a double Fourier series (11), calculating Φ_{mn} from Eq. (13), and substituting into Eqs. (14) and (15).

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Construction of a Liapunov Function in the Critical Case

J. R. Weiss* and E. Y. Yu†

Northwestern University, Evanston, Ill.

In this Note, we present a general method for constructing a Liapunov function for a linear system governed by the vector equation

$$dx(t)/dt = Ax(t) \tag{1}$$

where $x(t)$ is a n vector and A is a constant $n \times n$ matrix with distinct and purely imaginary eigenvalues. This system is Liapunov stable,[1] since A is a critical matrix as already

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* Graduate student, Department of Engineering Sciences.

† Associate Professor, Department of Mechanical Engineering and Astronautical Sciences. Member AIAA.

specified. It appears that one does not necessarily have to construct a Liapunov function for such a system, since the eigenvalue analysis already answers the question of stability. Nevertheless, the eigenvalue analysis usually yields various stable regions in the physical parameter space. If a Liapunov function is obtained in a special way based upon a physical property of the system, it may not be general enough to represent all the stable regions. The purpose of this Note is to record a general mathematical method of constructing a Liapunov function so that the Liapunov analysis will always be in agreement with the eigenvalue analysis. To our knowledge, such a method is not found in the literature.

Denote the n distinct eigenvalues of the matrix A by λ_i, $i = 1,2,\ldots,n$, and the corresponding eigenvectors by ϕ_i i.e., $A\phi_i = \lambda_i\phi_i$. Introduce the new basis $\{a_i\}_{i=1}^{n}$,

$$a_1 = \frac{\phi_1 + \phi_2}{2},\ a_2 = \frac{\phi_1 - \phi_2}{2i},\ldots,\ a_{n-1} = \frac{\phi_{n-1} + \phi_n}{2},\quad a_n = \frac{\phi_{n-1} - \phi_n}{2i} \tag{2}$$

In this new basis, A has the form

$$A = i\begin{vmatrix} 0 & -\lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \ldots & 0 & 0 \\ 0 & 0 & 0 & -\lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0\ . & 0 & 0 & 0 & 0 & .\ -\lambda_{n-1} \\ & & & \cdots & & \\ 0 & 0 & 0 & 0 & \lambda_{n-1} & 0 \end{vmatrix} \tag{3}$$

Now, define a function

$$v(x) = (Bx)^T\bar{x} \tag{4}$$

where B is a symmetric $n \times n$ matrix, the superscript T denotes the transpose, and the bar denotes the complex conjugate. If $v(x)$ is positive-definite and $dv(x)/dt$ is nonpositive along solutions of Eq. (1), then $v(x)$ is a Liapunov function.[1] Upon differentiating Eq. (4) with respect to t, it is found that $dv(x)/dt$ will be nonpositive if B satisfies the Liapunov equation

$$A^TB + BA = -C \tag{5}$$

for a non-negative matrix C. One solution to Eq. (5) written in the basis defined in Eq. (2) is that B equals the identity matrix and C equals the zero matrix, since A as given in Eq. (3) is skew-symmetric. Then the Liapunov function becomes

$$v(x) = x^T\bar{x} \tag{6}$$

where x is in the basis $\{a_i\}_{i=1}^{n}$.

The foregoing method can be used in the stability analysis of some dynamical systems. In particular, we will now apply it to the classical stability problem of the attitude motion of a satellite orbiting a spherical Earth in a circular orbit at a constant angular speed Ω with no disturbing torques. Such a problem was first studied by Lagrange,[2] in recent time by DeBra and Delp,[3] and by others. Let the Earth pointing rotating coordinate system be denoted by $O\text{-}X_1X_2X_3$, where OX_1 is along the outward local vertical, OX_2 is in the direction of the orbital velocity, and OX_3 is normal to the orbital plane. Let $O\text{-}x_1x_2x_3$ be the body coordinate system along the principal axes of the satellite. The orientation of the body coordinate system is obtained by a series of counter-clockwise rotations: first of an angle θ_1 about the OX_1 axis, then of an angle θ_2 about the OX'_2 axis, and finally of an angle θ_3 about the Ox_3 axis. In the linearized equations of motion, which were first obtained by Lagrange, the equation for the pitch angle θ_3 is decoupled from the equations for the yaw and roll angles θ_1 and θ_2. DeBra and Delp found by means of eigenvalue analysis of the linearized equations that the motion is stable if the satellite has either the configuration, $I_3 > I_2 > I_1$, or the configuration $I_2 > I_1 > I_3$, where I_1, I_2, and I_3 are the moments of inertia of the satellite about the $O\text{-}x_1x_2x_3$ axes, respectively. These two configurations correspond to the Lagrange and Delp regions,[3] respectively, in the k_1k_2-parameter space, where $k_1 = -(I_2 - I_3)/I_1$ and $k_2 = -(I_3 - I_1)/I_2$. It can be shown that the Hamiltonian of the system is a Liapunov function for the Lagrange region but not for the Delp region. The Lagrange equations for the roll and yaw motions can be transformed to a fourth-order system of the form of Eq. (1) on the vector $x = (\dot{\theta}_2,\theta_2',\theta_1,\theta_1')^T$ with the constant matrix A being $1 = d/d\,(\Omega\nu)$

$$A = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 4k_2 & 0 & 0 & -(k_2+1) \\ 0 & 0 & 0 & 1 \\ 0 & (1-k_1) & -k_1 & 0 \end{vmatrix} \tag{7}$$

The eigenvalues of this matrix A are

$$\begin{aligned} \lambda_1 &= i\{\tfrac{1}{2}(1 - 3k_2 - k_1k_2) - \tfrac{1}{2}[(1 - 3k_2 - k_1k_2)^2 + 16k_1k_2]^{1/2}\}^{1/2},\ -\lambda_1 \\ \lambda_3 &= i\{\tfrac{1}{2}(1 - 3k_2 - k_1k_2) + \tfrac{1}{2}[(1 - 3k_2 - k_1k_2)^2 + 16k_1k_2]^{1/2}\}^{1/2},\ -\lambda_3 \end{aligned} \tag{8}$$

In terms of the parameters b and c defined by

$$b = (4k_2 - \lambda_3^2)/(k_2 + 1) \text{ and } c = (4k_2 - \lambda_1^2)/(k_2 + 1)$$

the eigenvectors of A in Eq. (7) are

$$\phi_1 = \begin{vmatrix} 1 \\ \lambda_1 \\ c/\lambda_1 \\ c \end{vmatrix},\ \phi_2 = \bar{\phi}_1,\ \phi_3 = \begin{vmatrix} 1 \\ \lambda_3 \\ b/\lambda_3 \\ b \end{vmatrix},\ \phi_4 = \bar{\phi}_3$$

and the elements of the new basis a_i, $i = 1$–4, are the combinations given in Eq. (2). The vector x is then written in the new basis. According to Eq. (6), the Liapunov function for the fourth-order linear system governing the roll and yaw motions is

$$\begin{aligned} v(\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) = &\left(\frac{-b\Omega\theta_2 + \dot{\theta}_1}{\Omega(c-b)}\right)^2 + \left(\frac{-ib\lambda_1\dot{\theta}_2 + i\Omega\lambda_1\lambda_3^2\theta_1}{\Omega(c\lambda_3^2 - b\lambda_1^2)}\right)^2 + \left(\frac{c\Omega\theta_2 - \dot{\theta}_1}{\Omega(c-b)}\right)^2 + \\ &\left(\frac{ic\lambda_3\dot{\theta}_2 - i\Omega\lambda_1^2\lambda_3\theta_1}{\Omega(c\lambda_3^2 - b\lambda_1^2)}\right)^2 \end{aligned} \tag{9}$$

As a check, it can easily be shown that $dv(x)/dt$ is zero. It should be noted that this form of $v(x)$ is obtained from the fact that λ_1 and λ_3 are pure imaginary, as required to begin with. This is true if $1 - 3k_2 - k_1k_2 > 0$, $k_1k_2 < 0$, and $(1 - 3k_2 - k_1k_2)^2 + 16k_1k_2 > 0$. Since these inequalities, together with that required for the stability of pitch motion, are just those that define the Lagrange and Delp regions of the parameter space, the function given in Eq. (9) is a Liapunov function for both of these regions.

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